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AUTHOR(S):

SHIMOMURA, HIROAKI; HIRAI, TAKESHI

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CITATION:

SHIMOMURA, HIROAKI ...[et al]. On Group Topologies on the Group of Diffeomorphisms.  
数理解析研究所講究録 1997, 1017: 104-115

ISSUE DATE:

1997-11

URL:

<http://hdl.handle.net/2433/61636>

RIGHT:

## On Group Topologies on the Group of Diffeomorphisms

HIROAKI SHIMOMURA and TAKESHI HIRAI

(Fukui University)

(Kyoto University)

下村 宏彰 (福井大)

平井 武 (京都大)

### Introduction.

Let  $M$  be a connected, non-compact,  $\sigma$ -compact  $C^r$ -manifold with  $1 \leq r \leq \infty$ . Denote by  $\text{Diff}(M)$  the group of all diffeomorphisms and by  $\text{Diff}_0(M)$  its subgroup consisting of diffeomorphisms with compact supports. Here we study group topologies on the group  $G = \text{Diff}_0(M)$ . Usually, as seen in the beginning of [Ki], we have been considering on  $G$  the topology  $\tau$  given by the following way of convergence: *a sequence  $g_k, k = 1, 2, \dots$ , converges to  $g$  if supports of  $g$  and of all  $g_k$  are contained in a compact subset  $K$  and  $g_k \rightarrow g$  on  $K$  uniformly together with all derivatives.*

This topology  $\tau$  is normally understood as an inductive limit of topologies of canonical subgroups  $G_n \nearrow G, n \rightarrow \infty$ , as follows. First take an increasing sequence  $M_0 \subset M_1 \subset M_2 \subset \dots$  of relatively compact open subsets so that  $\bigcup_{n=0}^{\infty} M_n = M$  and that each  $K_n := \bar{M}_n$ , the closure of  $M_n$ , is a manifold with boundary. Put

$$G_n = \text{Diff}(K_n) := \{g \in G; \text{supp}(g) \subset K_n\}.$$

Then we have an increasing sequence of subgroups as

$$G_0 \subset G_1 \subset G_2 \subset \dots, \quad \bigcup_{n=0}^{\infty} G_n = G.$$

The topology  $\tau_n$  on  $G_n$  is given by considering  $G_n$  as a topological subgroup of the Fréchet Lie group  $\text{Diff}(M_n'')$ , where  $M_n''$  is the compact manifold obtained by patching  $M_n$  and its mirror image  $M_n'$  through the boundary. For the Lie group structure of the group  $\text{Diff}(M)$  of a compact manifold  $M$ , we refer [Le] or [Om]. When  $M = \mathbb{R}^d$  and  $M_n = \{x \in \mathbb{R}^d; \|x\| < n\}$ , the topology  $\tau_n$  is nothing but the uniform convergence of  $g_k \in G_n$  and also of all derivatives as  $k \rightarrow \infty$ .

In an algebraic sense,  $G = \lim_{n \rightarrow \infty} G_n$ , and as a topology on  $G$ , we have  $\tau = \lim_{n \rightarrow \infty} \tau_n$ . Since we will consider other topologies on  $G$  later, we denote this inductive limit topology as  $\tau_{ind}$ .

On the other hand, as Tatsuuma[Ta] proved, when a consistent increasing sequence of topological groups  $(G_n, \tau_n)$ , with a group topology  $\tau_n$  on  $G_n$ , is given, the inductive limit  $\tau_{ind}$  of topologies  $\tau_n$  is not necessarily a group topology, that is, it does not necessarily make the inductive limit group  $G = \lim_{n \rightarrow \infty} G_n$  a topological group. This negative result is contrary to the affirmative statement in [Iw, Article 75] or in [Enc, Article 210]. In fact, he gave a counter example even in a case of simple abelian groups (Example 1.1).

It seems for us that this phenomenon is rather general for the case of non-locally-compact topological groups.

In this paper, we prove that this is the case for diffeomorphism group  $\text{Diff}_0(M)$  for any non-compact  $M$ . Thus our main theorem here is the following.

**Theorem A.** *Let  $M$  be a connected, non-compact,  $\sigma$ -compact  $C^r$ -manifold,  $1 \leq r \leq \infty$ . For the group  $\text{Diff}_0(M)$ , the product map  $G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G$ , is not continuous with respect to the inductive limit topology  $\tau_{ind}$  on  $G$ .*

This fact does not affect so much the theory of unitary representations of the group  $G$ , because we can take, as our background, the topology  $\tau_{p.d.}$  on  $G$  which is defined by means of the set of  $\tau_{ind}$ -continuous positive definite functions (cf. §1). However it has certainly some effects, for instance, for determining *continuous* 1-cocycles  $\alpha(g, p)$ ,  $(g, p) \in G \times M$ , depending on which continuity we choose (cf. [HS]).

Note that if a sequence  $g_k \in G, k = 1, 2, \dots$ , is  $\tau_{ind}$ -convergent to  $g \in G$ , then there exists a compact subset  $K$  of  $M$  such that  $\text{supp}(g_k)$  and  $\text{supp}(g)$  are contained in  $K$ , and the convergence is as in [Ki]. To see this last assertion, we remark that the restriction on  $G_n = \text{Diff}(K_n)$  of the inductive limit  $\tau_{ind}$  on  $G$  is exactly the original  $\tau_n$ . In fact, let  $O_n$  be a  $\tau_n$ -open subset of  $G_n$ , then, for  $k > n$ , we can choose inductively a  $\tau_k$ -open subset  $O_k$  of  $G_k$  such that  $O_k \cap G_{k-1} = O_{k-1}$ , since the restriction of  $\tau_k$  onto  $G_{k-1}$  is equal to  $\tau_{k-1}$ . Put  $O = \bigcup_{k=n}^{\infty} O_k$ , then  $O$  is  $\tau_{ind}$ -open in  $G$  and  $O \cap G_n = O_n$ .

**Acknowledgements.** The authors express their thanks to Professors N. Tatsuuma and A. Yamasaki for their kind discussions and suggestions about many subjects containing especially inductive limits of group topologies.

Here, on this opportunity, the authors together express their deepest thanks to Professor Yasuo Yamasaki for his long acquaintance with us through our studies.

## §1. Some generalities on inductive limits.

Let us consider an inductive system  $G_\alpha (\alpha \in A), \psi_{\alpha\beta} (\alpha, \beta \in A, \alpha \preceq \beta)$ , of

topological groups, where  $A$  is a directed set and  $\psi_{\alpha\beta} : G_\alpha \rightarrow G_\beta$ , are injective continuous homomorphisms. Put  $G = \varinjlim G_\alpha$  and we identify each  $G_\alpha$  with its image in  $G$  through  $\psi_{\alpha\beta}$ 's. Denote by  $\tau_\alpha$  the group topology on  $G_\alpha$  and by  $\tau_{ind} = \varinjlim \tau_\alpha$  their inductive limit. Note that, by definition, a subset  $U$  of  $G$  is open with respect to  $\tau_{ind}$  (or  $\tau_{ind}$ -open in short) if and only if  $U \cap G_\alpha$  is  $\tau_\alpha$ -open in  $G_\alpha$  for any  $\alpha \in A$ .

We see easily the following fact on  $\tau_{ind}$ .

**Lemma B.** *On the inductive limit group  $G = \varinjlim G_\alpha$ , the following maps are continuous with respect to  $\tau_{ind} = \varinjlim \tau_\alpha$ :*

- (i) *the inverse:  $G \ni g \mapsto g^{-1} \in G$ ;*
- (ii) *the left and right translations: for a fixed  $h \in G$ ,*

$$G \ni g \mapsto gh \in G, \quad G \ni g \mapsto hg \in G.$$

However the product map  $G \times G \ni (g_1, g_2) \mapsto g_1 g_2 \in G$  is not necessarily  $\tau_{ind}$ -continuous as the following example of Tatsuuma shows.

**Example 1.1**([Ta]). Let  $G_n = \mathbf{Q} \times F^n$ ,  $F = \mathbf{R}$  or  $\mathbf{Q}$  with the usual non-discrete topology, and imbed  $G_n$  into  $G_{n+1}$  as  $x \mapsto (x, 0)$ . Then, for  $G = \varinjlim_{n \rightarrow \infty} G_n = \mathbf{Q} \times \prod' \mathbf{R}$ , the product map is not  $\tau_{ind}$ -continuous. Or, there exists an open neighbourhood  $U$  of the identity element  $e$  of  $G$  such that  $V^2$  is not contained in  $U$  for any open neighbourhood  $V$  of  $e$ .

Note that, if a sequence  $g_k \in G, k = 1, 2, \dots$ , converges to  $e$ , then there exists a  $G_n$  such that  $g_k \in G_n$  for all  $k$ , and they converge in  $G_n$ .

He also proved the following affirmative fact.

**Proposition C**([Ta]). *For an inductive sequence  $(G_n, \tau_n), n = 1, 2, \dots$ , of topological groups, assume that all  $G_n$ 's are locally compact. Then the inductive limit topology  $\tau_{ind} = \varinjlim \tau_n$  gives a group topology on  $G = \varinjlim_{n \rightarrow \infty} G_n$ .*

**Example 1.2**([Ya]). Let  $GL(\infty, F)$  with  $F = \mathbf{R}$  or  $\mathbf{C}$  be the inductive limit group of  $G_n = GL(n, F), n = 1, 2, \dots$ , where  $G_n$  is imbedded into  $G_{n+1}$  as

$$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, by the above proposition,  $\tau_{ind}$  is a group topology on  $GL(\infty, F)$ . A basis for  $\tau_{ind}$ -neighbourhoods of  $e$  is given by A.Yamasaki. Rewriting it in a different form, we get another basis as follows. For  $g \in GL(\infty, F)$ , put  $g = 1 + x, x =$

$(x_{ij})_{i,j=1}^\infty$ . Take  $\kappa = (\kappa_{ij})_{i,j=1}^\infty$ , with  $\kappa_{ij} > 0$ , and put

$$V(\kappa) = \{g = 1 + x; |x_{ij}| < \kappa_{ij} (\forall i, j)\}.$$

**Note 1.3.** Generally speaking, why  $\tau_{ind}$  does not give a group topology is that  $\tau_{ind}$  has too many open neighbourhoods of  $e$ . So we should have some criterion to decrease the number of these neighbourhoods. In this context, we can refer the case of locally convex topological vector spaces. In that case the criterion is the convexity of neighbourhoods.

As a group topology on  $G$  weaker than  $\tau_{ind}$ , one can propose the topology  $\tau_{p.d.}$  defined by means of the set  $\mathcal{P}(\tau_{ind})$  of all positive definite functions on  $G$  continuous with respect to  $\tau_{ind}$ . Note that a positive definite function  $f$  is  $\tau_{ind}$ -continuous on  $G$  if it is  $\tau_{ind}$ -continuous at  $e$ , because the topology  $\tau_{ind}$  is translation-invariant (by Lemma B(ii)), and the positive definiteness of  $f$  gives  $f(e) \geq |f(g)|$ ,  $f(g^{-1}) = \text{Conj}\{f(g)\}$ , and Krein's inequality [Kr]

$$|f(g) - f(h)|^2 \leq 2f(e) \{f(e) - \Re(f(gh^{-1}))\} \quad (g, h \in G).$$

By definition, an open neighbourhood of  $e$  with respect to  $\tau_{p.d.}$  is given as follows. Take a finite number of  $f_j \in \mathcal{P}(\tau_{ind})$ ,  $1 \leq j \leq N$ , and an  $\epsilon > 0$ , then

$$U(f_1, f_2, \dots, f_N; \epsilon) = \{g \in G; |f_j(g) - f_j(e)| < \epsilon (\forall j)\}.$$

The topology  $\tau_{p.d.}$  is also defined as a weakest topology on  $G$  which makes all  $\tau_{ind}$ -continuous unitary representations continuous.

Finally we note that  $\mathcal{P}(\tau_{ind}) = \mathcal{P}(\tau_{p.d.})$ .

## §2. Preparation for the proof of Theorem A.

Let  $d = \dim M$ . To express  $G = \text{Diff}_0(M)$  as an inductive limit, we choose  $M_0 \subset M_1 \subset \dots \subset M_n \subset \dots$  under the following additional condition.

(Condition 1) There exists a coordinate neighbourhood  $(V_M, \iota_M)$  containing the closure  $\bar{M}_1$  and such that, with respect to a  $C^r$ -class Riemannian structure on  $M$ ,  $M_0$  and  $M_1$  are open balls with the common center, and further that, under the coordinate map  $\iota_M$ , the Riemannian structure is of the canonical form on  $M_1$ :

$$ds^2 = dp_1^2 + dp_2^2 + \dots + dp_d^2 \quad \text{for } p = (p_i)_{i=1}^d \in M_1 \xrightarrow{\iota_M} \mathbf{R}^d.$$

Denote by  $\rho(p, q)$  the distance of two points  $p, q \in M$ . We fix the origin  $\mathbf{O}$  of the coordinates on the boundary  $\partial(M_0)$  of  $M_0$ , and put  $\rho(p) = \rho(p, \mathbf{O})$ .

Let  $C^r(\bar{M}_0, M_1)$  denotes the set of all maps from  $\bar{M}_0$  into  $M_1$  which are restrictions on  $\bar{M}_0$  of  $C^r$ -maps from some open sets containing  $\bar{M}_0$  into  $M_1$ . Take  $\phi \in C^r(\bar{M}_0, M_1)$ . For  $1 \leq k \leq r$ , finite, and  $p \in \bar{M}_0$ , put alike a jet at  $p$

$$j_p^k \phi = (\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \phi(p))_{|\alpha| \leq k},$$

with  $\partial_i = \frac{\partial}{\partial p_i}$ ,  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ .

Considering this value as an element of a Euclidean space  $(\mathbf{R}^d)^{N_k}$  for an appropriate  $N_k$ , we take its norm:

$$\|j_p^k \phi\| := \left( \sum_{|\alpha| \leq k} \|\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \phi(p)\|^2 \right)^{1/2},$$

and put for  $\phi, \psi \in C^r(\bar{M}_0, M_1) \subset C^r(\bar{M}_0, \mathbf{R}^d)$ ,

$$d^k(\phi, \psi) := \sup_{p \in \bar{M}_0} \|j_p^k(\phi - \psi)\|.$$

We put also, taking the  $k$ -th homogeneous part,

$$j_p^{(k)} \phi := (\partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} \phi(p))_{|\alpha|=k}, \quad d^{(k)}(\phi, \psi) := \sup_{p \in \bar{M}_0} \|j_p^{(k)}(\phi - \psi)\|.$$

The next lemma is a key of our proof of Theorem A. Let  $D_1, D_2 \subset \mathbf{R}^d$  be connected open sets, and  $C^r(D_1, D_2)$  be the set of all  $C^r$ -class maps  $\phi$  from  $D_1$  to  $D_2$ . For  $\phi = (\phi_i)_{i=1}^d \in C^r(D_1, D_2)$ , we have  $j_p^{(1)} \phi = (\partial_j \phi_i)_{1 \leq i, j \leq d}$ . Considering it as a linear map on  $\mathbf{R}^d$  canonically, we denote its operator norm by  $\|j_p^{(1)} \phi\|_{op}$ , where we take  $\|x\| = (x_1^2 + x_2^2 + \cdots + x_d^2)^{1/2}$  as the norm of  $x = (x_i)_{i=1}^d \in \mathbf{R}^d$ .

**Lemma 2.1.** *Let  $D \subset \mathbf{R}^d$  be an open ball and denote by  $\text{id}$  the identity map on  $D$ . Assume for  $\phi \in C^r(D, D)$ , the support  $\text{supp}(\phi) := \text{Cl}\{p \in D_1; \phi(p) \neq p = \text{id}(p)\}$  is compact, and*

$$\|j_p^{(1)}(\phi - \text{id})\|_{op} = \|j_p^{(1)} \phi - 1_d\|_{op} < 1 \quad (\forall p \in D),$$

where  $1_d$  denotes the  $d \times d$  identity matrix. Then  $\phi$  is a diffeomorphism on  $D$ .

*Proof.* Since  $\det(j_p^{(1)} \phi) \neq 0$  ( $\forall p \in D$ ), by the theorem of implicit functions, we see that  $\phi$  is an open map and locally diffeomorphic.

On the other hand,  $\phi$  is globally 1-1. In fact, for  $p, q \in D \subset \mathbf{R}^d$ , take  $p - q \in \mathbf{R}^d$  and put  $p_t = q + t(p - q)$  ( $0 \leq t \leq 1$ ), then

$$\phi(p) - \phi(q) = \int_0^1 \frac{d}{dt} \phi(p_t) dt = \int_0^1 (j_{p_t}^{(1)} \phi) (p - q) dt.$$

From the similar formula for  $\psi = \phi - \text{id}$ , we have

$$\| \psi(p) - \psi(q) \| \leq \int_0^1 \| j_{p_t} \psi \|_{op} \| p - q \| dt < \| p - q \|.$$

Hence  $\| \phi(p) - \phi(q) \| \geq \| p - q \| - \| \psi(p) - \psi(q) \| > 0$ .

Now let us prove that  $\phi$  is onto. To do so, it is enough to prove that  $\phi(D)$  is relatively closed, i.e.,  $D \cap \text{Cl}(\phi(D)) = \phi(D)$ , because we know already that  $\phi(D)$  is open. Here  $\text{Cl}(\phi(D))$  denotes the closure of  $\phi(D)$  in  $\mathbb{R}^d$ . Take a  $p \in D \cap \text{Cl}(\phi(D))$ . Then there exists a sequence  $q_n \in D$  such that  $\phi(q_n) \rightarrow p$  as  $n \rightarrow \infty$ . Since  $\phi$  is 1-1 and  $= \text{id}$  near the boundary  $\partial(D)$ ,  $q_n$  has an accumulation point  $q$  inside  $D$ . Thus we get  $p = \phi(q)$ . Q.E.D.

### §3. Behavior of a diffeomorphism on $M_0$ and $\bar{M}_0$ .

**3.1. A basis of neighbourhoods of  $e \in G_0$ .** We denote the identity map  $\text{id}$  on  $M$  also by  $e$ , since it is the identity element of  $G$ . Put

$$\Omega = \{ g \in G; g\bar{M}_0 \subset M_1 \} \subset G.$$

Then  $\Omega$  is  $\tau_{ind}$ -open in  $G$ , as is easily seen. Note that, for  $g \in \Omega$ , its restriction  $g|_{\bar{M}_0}$  on  $\bar{M}_0$  belongs to  $C^r(\bar{M}_0, M_1)$ .

We define subsets  $W_k$  of  $\Omega$  as follows depending on the class  $C^r$ :

$$\begin{aligned} W_k &:= \{ g \in \Omega; d^k(g, e) \leq 1/k \} \quad \text{in Case } r = \infty, \\ W_k &:= \{ g \in \Omega; d^r(g, e) \leq 1/k \} \quad \text{in Case } r < \infty. \end{aligned}$$

Then we have the following lemma.

**Lemma 3.1.** *Put  $W_{k,0} := W_k \cap G_0$  for  $k = 1, 2, \dots$ . Then they form a basis of neighbourhoods of the identity element  $e \in G_0$  with respect to the topology  $\tau_0$ .*

**3.2. Convex combination of maps.** Take  $g \in \Omega$ . For  $0 \leq s \leq 1$ , we can put

$$(3.1) \quad g_s := s \cdot \text{id}_{\bar{M}_0} + (1-s) \cdot g|_{\bar{M}_0} \in C^r(\bar{M}_0, M_1).$$

More generally we put, for  $\phi \in C^r(\bar{M}_0, M_1)$ ,

$$\phi_s := s \cdot \text{id}_{\bar{M}_0} + (1-s) \cdot \phi \in C^r(\bar{M}_0, M_1).$$

Further put

$$\begin{aligned} \alpha_k(\phi) &:= \inf \{ s; 0 \leq s \leq 1, d^k(\phi_s, \text{id}) \leq 1/k \} \quad \text{in Case } r = \infty, \\ \alpha_k(\phi) &:= \inf \{ s; 0 \leq s \leq 1, d^r(\phi_s, \text{id}) \leq 1/k \} \quad \text{in Case } r < \infty. \end{aligned}$$

Since  $d^k(\phi_s, \text{id}) = \sup_{p \in \bar{M}_0} \|j_p^k(\phi_s - \text{id})\| = (1-s) \cdot d^k(\phi, \text{id})$ , we have according as  $r = \infty$  or  $r < \infty$ ,

$$(3.2) \quad \alpha_k(\phi) = 0 \vee \left(1 - \frac{1}{k \cdot d^k(\phi, \text{id})}\right) \quad \text{in Case } r = \infty,$$

$$(3.2') \quad \alpha_k(\phi) = 0 \vee \left(1 - \frac{1}{k \cdot d^r(\phi, \text{id})}\right) \quad \text{in Case } r < \infty.$$

Define further, for  $\phi \in C^r(\bar{M}_0, M_1)$ ,

$$P_k \phi = \phi_{\alpha_k(\phi)} = \alpha_k(\phi) \cdot \text{id}_{\bar{M}_0} + (1 - \alpha_k(\phi)) \cdot \phi \in C^r(\bar{M}_0, M_1).$$

Then we have the following facts.

( $\nabla$ ) Let  $g \in W_k \subset \Omega$ . Then  $\alpha_k(g) = 0$ , whence  $P_k g = g|_{\bar{M}_0}$ .

( $\square$ ) Let  $g \in G_0 \subset \Omega$ . Assume  $g \in W_{k,0} = W_k \cap G_0$  with  $k \geq 2$ . Then, for any  $s, 0 \leq s \leq 1$ , we can extend  $g_s$  outside of  $M_0$  as  $g_s = \text{id}$ , and get  $g_s \in G_0 \subset G$ .

*Proof.* Since  $M_0$  is an open ball, we have  $g_s \in C_0^r(M_0, M_0)$ . Moreover, for any  $p \in M_0$ ,

$$\|j_p^{(1)}(g_s - \text{id})\|_{op} \leq d^{(1)}(g_s, \text{id}) \leq d^1(g_s, \text{id}) \leq 1/k < 1.$$

By Lemma 2.1 applied to  $D = M_0$ , we see  $g_s \in \text{Diff}_0(M_0) \subset G_0 \subset G$ .

**3.3. A crucial inequality on  $M_0$ .** Now put for  $g \in \Omega$

$$(3.3) \quad \beta_k := \inf_{g \in W_{k,0}} \int_{\bar{M}_0} \rho(g(p)) dp = \inf_{g \in W_{k,0}} \int_{\bar{M}_0} \|g(p)\| dp_1 dp_2 \cdots dp_d,$$

where  $p = (p_i)_{i=1}^d$ ,  $dp = dp_1 dp_2 \cdots dp_d$ , and  $\|g(p)\| = (\sum_{i=1}^d g_i(p)^2)^{1/2}$  with  $g(p) = (g_i(p))_{i=1}^d$ .

The inequality in the following lemma reflects the fact that  $G_0$  is not locally compact and is crucial for our proof of Theorem A.

**Lemma 3.2.** *Let  $k \geq 2$ . Then, for any  $g \in W_{k,0} = W_k \cap G_0$ , we have*

$$\int_{\bar{M}_0} \rho(g(p)) dp > \beta_k.$$

*Proof.* STEP 1. Since  $g \in G_0$ ,  $\text{supp}(g) \subset \bar{M}_0$  and so  $g$  and the identity map  $\text{id}$  have, at the origin  $O$ ,  $C^r$ -class contact. Hence

$$j_O^{k'}(g) = j_O^{k'}(\text{id}) \quad (\forall k' \leq r, \text{finite}).$$



We can consider  $g - \text{id}$  as an element of  $C^r(M_1, \mathbf{R}^d)$ , then

$$j_{\mathbf{O}}^{k'}(g - \text{id}) = 0 \quad (\forall k' \leq r, \text{finite}).$$

We fix  $k \geq 2$ , and take  $k' = k$  in Case  $r = \infty$ , and  $k' = r$  in Case  $r < \infty$ . Then there exists an open neighbourhood  $U_M$  of  $\mathbf{O}$  in  $M$  such that

$$\begin{aligned} \|j_p^{k'}(g - \text{id})\| &< \frac{1}{2k} \quad (\forall p \in U_M \cap M_0), \\ j_p^{k'}(g - \text{id}) &= \mathbf{0} \quad (\forall p \notin M_0). \end{aligned}$$

Now take an  $\eta = (\eta_i)_{i=1}^d \in C_0^r(U_M \cap M_0, \mathbf{R}^d)$  satisfying

$$(3.4) \quad \|j_p^{k'}\eta\| < \frac{1}{2k} \quad \text{and} \quad \|j_p^0\eta\| = \|\eta\| < \text{diam}(M_1) - \text{diam}(M_0),$$

where  $\text{diam}(M_1)$  denotes the diameter of  $M_1$ . Put  $\phi = g - \eta$ . Then,

$$\begin{aligned} \phi(\bar{M}_0) &\subset M_1 \quad \text{and} \quad \phi = \text{id} \quad \text{on} \quad M_1 \setminus M_0, \\ \|j_p^{k'}(\phi - \text{id})\| &< \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \quad (\forall p \in U_M \cap M_0). \end{aligned}$$

Hence  $\phi \in C^r(M_1, M_1)$  and, for any  $p \in M_1$ ,

$$\|j_p^{(1)}(\phi - \text{id})\|_{op} \leq \|j_p^{k'}(\phi - \text{id})\| < \frac{1}{k} < 1.$$

Therefore we can apply Lemma 2.1 to  $\phi$  and  $D = M_1$ , and see that  $\phi \in \text{Diff}_0(M_1)$ . Since  $\text{supp}(\phi) \subset \bar{M}_0$ , we get  $\phi \in G_0 = \text{Diff}(\bar{M}_0)$  and so  $\phi \in W_{k,0} = W_k \cap G_0$ .

STEP 2. Let us compare the following two values:

$$\begin{aligned} A &:= \int_{\bar{M}_0} \rho(g(p)) dp = \int_{\bar{M}_0} \left( \sum_{i=1}^d g_i(p)^2 \right)^{1/2} dp, \\ B &:= \int_{\bar{M}_0} \rho(\phi(p)) dp = \int_{\bar{M}_0} \left( \sum_{i=1}^d (g_i(p) - \eta_i(p))^2 \right)^{1/2} dp. \end{aligned}$$

To get  $A > B$  ( $\geq \beta_k$ ), it is sufficient to have the following:

$$\begin{aligned} |g_i(p)| &\geq |g_i(p) - \eta_i(p)| \quad (\forall i, \forall p \in \bar{M}_0), \\ |g_{i_0}(p_0)| &> |g_{i_0}(p_0) - \eta_{i_0}(p_0)| \quad (\exists i_0, \exists p_0 \in \bar{M}_0), \end{aligned}$$

On the other hand, since the maps  $g$  and  $\text{id}$  are sufficiently near to each other on  $U_M \cap M_0$ , there certainly exist  $i_0$  and  $p_0 \in U_M \cap M_0$  such that  $g_{i_0}(p_0) \neq 0$ .

Then there exists a small neighbourhood  $U(p_0)$  of  $p_0$  such that, for  $\epsilon = 1$  or  $-1$  and some  $\kappa > 0$ ,

$$\epsilon \cdot g_{i_0}(p) > \kappa \quad (\forall p \in U(p_0)).$$

We can choose  $\eta = (\eta_i)_{i=1}^d$  in such a way that  $\eta_i = 0$  for  $i \neq i_0$ , and  $\eta_0 \in C_0^r(U(p_0) \cap U_M \cap M_0, \mathbf{R}^d)$  satisfies the condition (3.4) and

$$\epsilon \cdot \eta_{i_0}(p_0) > 0, \quad \kappa \geq \epsilon \cdot \eta_{i_0}(p) \geq 0 \quad (\forall p).$$

Under this choice of  $\eta$  the above sufficient condition for  $A > B$  holds.

This gives that  $A > \beta_k$ , which is to be proved. Q.E.D.

#### §4. A $\tau_{ind}$ -neighbourhood of $e \in G$ .

**4.1. Neighbourhood  $U$ .** We define a  $\tau_{ind}$ -neighbourhood  $U$  of  $e \in G$ , for which it will be proved that  $V^2 \not\subset U$  for any  $\tau_{ind}$ -neighbourhood  $V$  of  $e \in G$ .

Let  $M_0^c = M \setminus M_0$ , and put, for  $g \in \Omega \subset G$ ,

$$(4.1) \quad F_k(g) := \left| \int_{\bar{M}_0} \rho((P_k g)(p)) dp - \beta_k \right| + \int_{M_0^c} \rho(g(p), \text{id}(p)) dp.$$

where  $\text{id}(p) = p$ . Then the following fact is a consequence of Lemma 3.2.

**Lemma 4.1.** *Let  $k \geq 2$ . Then,  $F_k(g) > 0$  ( $\forall g \in \Omega$ ).*

*Proof.* Assume that the 2nd term in  $F_k(g)$  is equal to zero. Then,  $g = \text{id}$  on  $M_0^c$ , and so  $\text{supp}(g) \subset \bar{M}_0$  whence  $g \in G_0 \subset C^r(\bar{M}_0, M_1)$ . Then,

$$\begin{aligned} P_k g &\in C^r(\bar{M}_0, M_1) \subset C^r(M_1, M_1), \\ \text{supp}(P_k g) &\subset \text{supp}(g) \subset \bar{M}_0 \quad \text{and} \quad d^{k'}(P_k g, \text{id}) \leq 1/k < 1, \end{aligned}$$

where  $k' = k$  or  $= r$  according as  $r = \infty$  or  $r < \infty$ . Therefore we can apply Lemma 2.1 to  $\phi = P_k g$  and  $D = M_1$ , and see that  $P_k g \in \text{Diff}(\bar{M}_0) = G_0$ . Then by Lemma 3.2 we get

$$\int_{\bar{M}_0} \rho((P_k g)(p)) dp > \beta_k.$$

This means that the 1st term in (3.4) of  $F_k(g)$  is positive, and so  $F_k(g) > 0$ .

**4.2. Proof of Theorem A.** Choose non-empty open sets  $O_k$  in such a way that  $O_k \subset M_k \setminus \bar{M}_{k-1}$  for  $k \geq 2$ . Fix  $\gamma > 1$ , and for  $k \geq 2$ , put

$$U_k := \left\{ g \in \Omega; F_k(g) > \gamma \cdot \int_{O_k} \rho(g(p), \text{id}(p)) dp \right\}.$$

Since  $G_n = \text{Diff}(\bar{M}_n) = \{g \in G; \text{supp}(g) \subset \bar{M}_n\}$ , we see that, if  $n < k$ , then  $g = \text{id}$  on  $O_k$ . Then, by Lemma 4.1,  $U_k \cap G_n = \Omega \cap G_n$ , and this is  $\tau_n$ -open in  $G_n$ . In particular,  $G_0 = \Omega \cap G_0 \subset U_k$ . Put

$$U = \bigcap_{k=2}^{\infty} U_k \subset \Omega.$$

**Lemma 4.2.** *The subset  $U$  is  $\tau_{\text{ind}}$ -open in  $G$ .*

*Proof.* For any  $n \geq 2$ , the intersection  $U \cap G_n$  is  $\tau_{\text{ind}}$ -open in  $G_n$ , because

$$U \cap G_n = \bigcap_{k=2}^n (U_k \cap G_n) \cap (\Omega \cap G_n).$$

Now we come to the final stage of the proof of Theorem A, and it is enough for us to prove the following lemma.

**Lemma 4.3.** *There does not exist any  $\tau_{\text{ind}}$ -neighbourhood  $V$  of  $e \in G$  such that  $V^2 \subset U$ .*

*Proof.* Suppose the contrary and let  $V$  be such that  $V^2 \subset U$ . Since  $V \cap G_0$  is  $\tau_0$ -open and  $W_{k,0}$ 's form a basis of  $\tau_0$ -neighbourhoods of  $e \in G_0$ , there exists a  $W_{k,0}$  such that  $V \cap G_0 \supset W_{k,0}$ . Put  $V_k = V \cap \text{Diff}_0(O_k)$ . Then

$$W_{k,0} V_k \subset V^2 \subset U \subset U_k \subset \Omega.$$

Hence, for any  $g \in W_{k,0}, h \in V_k$ ,

$$F_k(g \circ h) > \gamma \cdot \int_{O_k} \rho((g \circ h)(p), \text{id}(p)) dp.$$

Note that  $\text{supp}(g) \subset \bar{M}_0$ ,  $\text{supp}(h) \subset M_k \setminus M_{k-1}$ , and that

$$g \circ h = g \text{ on } \bar{M}_0, g \circ h = h \text{ on } O_k, g \circ h = \text{id} \text{ anywhere else.}$$

Hence

$$\left| \int_{\bar{M}_0} \rho((P_k g)(p)) dp - \beta_k \right| > (\gamma - 1) \cdot \int_{O_k} \rho(h(p), \text{id}(p)) dp.$$

Further, since  $g \in W_{k,0} = W_k \cap G_0$ , we have  $P_k g = g$ , and the above inequality turns out to be

$$\int_{\bar{M}_0} \rho(g(p)) dp - \beta_k > (\gamma - 1) \cdot \int_{O_k} \rho(h(p), \text{id}(p)) dp.$$

Taking the infimum over  $g \in W_{k,0}$ , we get 0 on the left hand side and so

$$0 = \int_{O_k} \rho(h(p), \text{id}(p)) dp.$$

Hence  $h = \text{id}$ . This means that  $V \cap \text{Diff}_0(O_k) = \{\text{id}\}$ . A contradiction.

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